Capacitated Network Design Games on a Generalized Fair Allocation Model*

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Abstract

The cost-sharing connection game is a variant of routing games on a network. In this model, given a directed graph with edge-costs and edge-capacities, each agent wants to construct a path from a source to a sink with low cost. The cost of each edge is shared by the users based on a cost-sharing function. One of simple cost-sharing functions is defined as the cost divided by the number of users. In fact, most of the previous papers about cost-sharing connection games addressed this cost-sharing function. It models an ideal setting, where no overhead arises when people share things, though it might be quite rare in real life; it is more realistic to consider the setting that the cost that an agent should pay is the original cost per the number of the agents plus the overhead. In this paper, we model the more realistic scenario of cost-sharing connection games by generalizing cost-sharing functions. Our generalization gives not a concrete generalized cost-sharing function but a class of cost-sharing functions satisfying the following natural properties: they are (1) non-increasing, (2) lower bounded by the original cost per the number of the agents, and (3) upper bounded by the original cost, which enables to represent various scenarios of cost-sharing. We investigate the Price of Anarchy (PoA) and the Price of Stability (PoS) of sum-cost and max-cost criteria under the generalized cost-sharing function. In spite of the generalization, we obtain the same bounds of PoA and PoS as the cost-sharing with no overhead except PoS of sum-cost, where the PoS of sum-cost increases from $\log n$ to n by the generalization. All the bounds that we give are tight. We further investigate the bounds from the viewpoints of graph classes, such as parallel-rinks graphs, series-parallel graphs, and directed acyclic graphs, which show critical differences of PoS/PoA values.

1 Introduction

The capacitated symmetric cost-sharing connection game (CSCSG) is a network design model of multiple agents' sharing costs to construct a network infrastructure for connecting a given source-sink pair. In the game, a possible network structure is given, but actual links are not built yet. For example, imagine to build an overlay network structure on a physical network. Each agent wants to construct a path from source s to sink t. To construct a path, each agent builds links by paying the costs associated with them. Two or more agents can commonly use a link if the number of agents is within the capacity associated with the link, and in such a case, the cost of the link is fairly shared by the agents that use it. Thus, the more agents use a common link, the less cost of the link they pay. Under this setting, each agent selfishly chooses a path to construct so that they minimize their costs to pay. The CSCSG is quite useful and can model many real-world situations for sharing the cost of a designed network, such as a virtual overlay, multicast tree, or other sub-network of the Internet. The CSCSG is firstly introduced by Feldman and Ron [8].

In the previous studies of CSCSG, the link cost is fairly shared, which means that the total cost paid for a link does not vary even if any number of agents use it. However, sharing resources yields more or less extra costs (overheads) in realistic cost-sharing situations; by increasing the number of users, extra commission fees are charged, service degradation occurs and so on. The existing models are not powerful enough to handle such situations.

In this paper, we model the more realistic scenario of CSCSG by generalizing cost-sharing functions. Our generalization gives not a concrete generalized cost-sharing function but a class of cost-sharing functions satisfying certain natural properties. Let p_e and c_e be

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the cost and capacity associated with link (edge) e, respectively. Suppose that x agents use link e, where $x \leq c_e$. In our model, a cost-sharing function $f_e(x)$ for link e is (1) non-increasing with respect to x, (2) $f_e(x) \geq p_e/x$, and (3) $f_e(1) = p_e$. Condition (1) is a natural property in cost-sharing models, (2) represents the situation that if two or more agents use a link, overheads may arises, and (3) represents that no overhead arises when only an agent uses the edge. Note that (2) implies that (2') the total cost paid by all the agents for e is at least p_e. Also note that by combining properties (1) and (3) we have (3') $f_e(x) \leq p_e$ for any positive integer x, which implies that the overhead is not too large and a cost paid by an agent is upper bounded by p_e; otherwise no one wants to cooperate. We emphasize that this significant generalization does not restrict any nature of fair cost-sharing. We believe that any natural fair cost-sharing function is in this scheme. Note that the cost-sharing function in the previous studies [8, 6, 7] is $f_e(x) = p_e/x$, which clearly satisfies (1), (2) and (3).

We investigate the Price of Anarchy (PoA) and the Price of Stability (PoS) of the game. A pure Nash equilibrium (we simply say Nash equilibrium) is a state where no agent can reduce its cost by changing the path that he/she currently chooses. Such a Nash equilibrium does not always exist in a general game, but it does in CSCSG. That is, a CSCSG converges to a Nash equilibrium. Thus, a major interest of analyzing games is to measure a goodness of Nash equilibrium. As social goodness measures, we consider two criteria. One is *sum-cost* criterion, where the social cost function is defined as the summation of the costs paid by all the agents, and the other is *max-cost* criterion, where it is defined as the maximum among the costs paid by all the agents. Both PoA and PoS are well used measures for evaluating the efficiency of Nash equilibria of games. The PoA is the ratio between the cost of the worst Nash equilibrium and the social optimum, whereas the PoS refers to the ratio between the cost of the best Nash equilibrium and the social optimum.

The previous studies also investigate PoA and PoS of these cost criteria under their game models. For details, see Section 1.2. In spite of the generalization, we obtain the same bounds of PoA and PoS as the cost-sharing with no overhead except PoS of sum-cost, where the PoS of sum-cost increases from $\log n$ to n by the generalization. All the bounds that we give for CSCSG are tight. We further investigate the bounds from the viewpoints of graph classes, such as parallel-rinks graphs, series-parallel graphs, and directed acyclic graphs, which show critical differences of PoS/PoA values. The details are summarized in Section 1.1.

1.1 Our contribution

In this paper, we investigate the PoA and the PoS of capacitated symmetric cost-sharing connection games under a generalized cost-sharing scheme as explained above. We address two criteria of the social cost: sum-cost and max-cost.

As for the sum-cost case, we first show that PoA is unbounded even on directed acyclic graphs (DAGs). On the other hand, on series-parallel graphs (SP graphs), we show that PoA under sum-cost is at most n and it is tight, that is, there is an example whose PoA is n. For PoS, we show that it is at most n and there is an example whose PoS under sum-cost is n. This gives the difference from the previous study, which shows that PoS is at most $\log n$ and it is tight with ordinary fair cost-sharing functions [8].

Next, we show the results on the max-cost. As with the sum-cost, we show that PoA under max-cost is unbounded on directed acyclic graphs. On SP graphs, we show that PoA is at most n and it is tight. We also show that PoS is at most n and it is tight. These results imply that the significant generalization does not affect PoA and PoS under max-cost.

We then discuss the capacitated *asymmetric* cost-sharing connection games where agents have different source and sink nodes. We observe that the lower bounds of PoA and PoS of CSCSG hold for the asymmetric case. Moreover, we show that PoS under sum-cost is at most n and PoS under max-cost is at most n^2.

As a final remark, our results presented in this paper are for directed graphs, but all the results except for directed acyclic cases can be easily modified to undirected cases by a standard transformation, for example. In the sense, our results are generic, which includes the results of [8, 6].

The results in this paper are summarized in Tables 1 and 2, respectively.

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*1 This is proved by using the potential function $\Phi(\mathbf{s}) = \sum_{e\in E}\sum_{x=1}^{x_e(\mathbf{s})} \frac{p_e}{x}$. Although Anshelevich et al. only gave the upper bound for the uncapacitated case, we can easily observe that it holds for the capacitated case.

*2 Feldman and Ron gave the lower bound for *undirected* parallel-link graphs in [8]. However, it can be modified to a *directed* parallel link graph.

Table 1. The summary of PoA and PoS of CSCSG under sum-cost criterion.

		parallel-link	series-parallel	DAG	General
Uncapacitated	PoA	n [1]			
	PoS	1 [1]			
Capacitated	PoA	n (**UB [Thm. 2]**, LB [1])		**unbounded [Thm. 1]**	
	PoS	$\log n$ (UB [1]^{*1}, LB [8]^{*2})			
Capacitated+General cost **[Our Setting]**	PoA	n **[Thm. 2]**		**unbounded [Thm. 1]**	
	PoS	n **[Thm. 3]**			

Table 2. The summary of PoA and PoS of CSCSG under max-cost criterion.

		parallel-link	series-parallel	DAG	General
Uncapacitated	PoA	n [1]			
	PoS	1 [1]			
Capacitated	PoA	n (**UB [Thm. 5]**, LB [8]^{*2})		**unbounded [Thm. 4]**	
	PoS	n (UB [6], LB [8]^{*2})			
Capacitated+General cost **[Our Setting]**	PoA	n **[Thm. 5]**		**unbounded [Thm. 4]**	
	PoS	n **[Thm. 6]**			

1.2 Related work

The cost-sharing connection game (CSG) is firstly introduced by [1]. Anshelevich et al. show that for *uncapacitated* cost-sharing connection game, the upper bound of PoA under sum-cost is at most n, and it is tight. For PoS, under sum-cost is 1. They also show that the PoS under sum-cost of every *asymmetric* CSG, where agents have different source and sink nodes, can be bounded by $\log n$. Epstein, Feldman and Mansour study the *strong equilibria* of cost-sharing connection games [5].

Feldman and Ron [8] introduce a capacitated variant of CSGs on undirected graphs. For the variant, they give the tight bounds of PoA and PoS under both sum-cost and max-cost for several graph classes except the PoS under max-cost for general graphs. Note that their results only holds for symmetric CSGs. Erlebach and Radoja fill the gap of the exception [6]. Feldman and Ofir investigate the strong equilibria for the capacitated version of CSGs [7].

In the literature of computing a Nash equilibrium, Anshelevich et al. prove that computing cheap Nash equilibria is NP-complete on CSGs [1]. Also, Vasilis show that finding a Nash equilibrium on a CSG is PLS-complete [13].

There are vast applications of CSGs. A natural application is the decision-making in sharing economy [1, 2, 3]. Radko and Laclau mention the relationship between CSGs and machine learning [12].

The previous studies for CSCSG do not consider any overhead, but when we share some resource (or tasks) it yields some overhead in general. In fact, controlling overheads to share tasks is a major issue in grid/parallel computing fields [9]. Furthermore, in the context of sharing economy, the transaction cost is considered a part of overheads [10].

2 Model

2.1 Capacitated Symmetric Cost-Sharing Connection Games

A capacitated symmetric cost-sharing connection game (CSCSG) Δ is a tuple:

$$\Delta = (n, G = (V, E), s, t, \{p_e\}_{e\in E}, \{c_e\}_{e\in E})$$

where n is the number of agents, $G = (V, E)$ is a directed graph, $s, t \in V$ are the *source* and *sink* nodes, $p_e \in \mathbb{R}^{\geq 0}$ is the cost of an edge e, and $c_e \in \mathbb{N}^{\geq 0}$ is the capacity of an edge e, i.e., the upper bound of the number of agents that can use edge e. An edge is also called a *link*. The purpose of each agent j is to construct an s-t path in G. An s-t path chosen by agent j is called a *strategy* of agent j, denoted by s_j. A tuple $\mathbf{s} = (s_1, \ldots, s_n)$ of strategies of n agents is called a *strategy profile*. We denote by $E(s_j) \subseteq E$ the set of edges in s-t path s_j. Namely, $E(s_j)$ is the set of edges used by agent j. Moreover, for a strategy profile $\mathbf{s} = (s_1, \ldots, s_n)$, we define $E(\mathbf{s}) = \bigcup_j E(s_j)$, which is the set of edges used in $\mathbf{s}$.

Let $x_e(\mathbf{s}) = |\{j \mid e \in E(s_j)\}|$ be the number of agents who use edge e in a strategy profile $\mathbf{s}$. A strategy profile $\mathbf{s}$ is said to be *feasible* if $\mathbf{s}$ satisfies $x_e(\mathbf{s}) \leq c_e$ for every $e \in E$. Furthermore, we say the game is *feasible* if there is at least one feasible strategy profile. In this paper, we deal with only feasible games, that is, games have at least one feasible strategy profile.

2.2 Cost-Sharing Function and Social Cost

In a general network design game, for a strategy profile $\mathbf{s}$, every agent j who uses an edge e should pay some cost based on a *payment* function $f_{e,j}(x_e(\mathbf{s}))$; agent j pays $\sum_{e \in E(s_j)} f_{e,j}(x_e(\mathbf{s}))$ in total. In cost sharing connection games, the cost imposed to an edge e is fairly divided into the agents using e; the cost paid by an agent using e is determined by a *cost-sharing function* $f_e(x_e(\mathbf{s}))$, and the total cost of agent j is

$$p_j(\mathbf{s}) = \begin{cases} \sum_{e \in E(s_j)} f_e(x_e(\mathbf{s})) & \forall e \in E(s_j), x_e(\mathbf{s}) \leq c_e \\ \infty & \text{otherwise} \end{cases}.$$

In this paper, we assume that a cost-sharing function $f_e(x)$ satisfies (1) *non-increasing*, (2) $f_e(x) \geq p_e/x$ and (3) $f_e(1) = p_e$.

We denote by (Δ, F) a CSCSG on Δ with the set of cost-sharing functions $F = \{f_e \mid e \in E\}$. To emphasize all functions in F satisfy the properties (1), (2) and (3), we say that F is in the generalized cost-sharing scheme, denoted by $\mathcal{F}^*$. Recall that (2) implies (2') the total cost paid by all the agents for e is at least p_e, and (3') $f_e(x) \leq p_e$ for any $x \geq 1$. If there is no overhead for sharing an edge e, the cost agent j pays for e under $\mathbf{s}$ is defined by $f_e(x_e(\mathbf{s})) = p_e/x_e(\mathbf{s})$. Let us denote $F_{\mathrm{ord}} = \{p_e/x_e(\mathbf{s}) \mid e \in E\}$, and clearly $F_{\mathrm{ord}} \in \mathcal{F}^*$. Previous studies such as [1, 5, 6, 7, 8] adopt this special case $(\Delta, F_{\mathrm{ord}})$.

We further denote by $\mathcal{F}_{all}$ the class of any type of payment functions, which include non-fair or even meaningless ones in the cost-sharing context. We introduce this class of functions just to contrast it with $\mathcal{F}^*$. For a class $\mathcal{F}$ of cost-sharing functions, we sometimes write $(\Delta, \mathcal{F})$ instead of writing "(Δ, F) for any $F \in \mathcal{F}$".

In CSCSGs, we consider two types of social costs for strategy profiles. The *sum-cost* of a strategy profile $\mathbf{s}$ is the total cost of all agents, that is, $\mathrm{cost}_{sc}(\mathbf{s}) = \sum_j p_j(\mathbf{s})$. The *max-cost* of a strategy profile $\mathbf{s}$ is the maximum among the costs paid by all the agents, that is, $\mathrm{cost}_{mc}(\mathbf{s}) = \max_j p_j(\mathbf{s})$.

2.3 The Existence of Nash Equilibrium

Given a strategy profile $\mathbf{s}$, if there is an agent j such that $p_j(\mathbf{s}) > p_j(s'_j, \mathbf{s}_{-j})$ for some s'_j, agent j has an incentive to change its strategy from s_j to s'_j. We call this type of change a *deviation*. A strategy profile $\mathbf{s}$ is called a *Nash equilibrium* if any agent does not have an incentive to deviate from $\mathbf{s}$, that is, $p_j(\mathbf{s}) \leq p_j(s'_j, \mathbf{s}_{-j})$ holds for any agent j and any s'_j, where s'_j is a new strategy of agent j and $\mathbf{s}_{-j} = \mathbf{s} \setminus \{s_j\}$ is the strategy profile $\mathbf{s}$ excluding s_j. We denote the set of Nash equilibria in CSCSG (Δ, F) by $\mathrm{NE}(\Delta, F)$.

In the proof of Theorem 2.1 of [1], it is shown that any non-capacitated network design game always has a Nash equilibrium by an argument using a potential function. Because we only consider feasible games, a similar argument can be applied to CSCSG (Δ, F) using the following potential function: $\Phi(\mathbf{s}) = \sum_{e \in E} \sum_{x=1}^{x_e(\mathbf{s})} f_e(x)$, where $f_e \in F$.

Proposition 1. *For any* CSCSG $(\Delta, \mathcal{F}_{all})$*, there exists a pure Nash equilibrium.*

2.4 Price of Anarchy and Price of Stability

The Price of Anarchy (PoA) and the Price of Stability (PoS) measure how inefficient the cost at a Nash equilibrium is for the optimal cost. Let $\mathbf{s}^*_{sc}$ be an optimal strategy profile under sum-cost, and $\mathbf{s}^*_{mc}$ be an optimal strategy profile under max-cost, respectively. Then the PoA's of (Δ, F) under sum-cost and max-cost are defined as follows.

$$\mathrm{PoA}_{sc}(\Delta, F) = \frac{\max_{s \in \mathrm{NE}(\Delta, F)} \mathrm{cost}_{sc}(\mathbf{s})}{\mathrm{cost}_{sc}(\mathbf{s}^*_{sc})}$$

$$\mathrm{PoA}_{mc}(\Delta, F) = \frac{\max_{s \in \mathrm{NE}(\Delta, F)} \mathrm{cost}_{mc}(\mathbf{s})}{\mathrm{cost}_{mc}(\mathbf{s}^*_{mc})}$$

Similarly, the PoS's of (Δ, F) under sum-cost and max-cost are defined as follows.

$$\mathrm{PoS}_{sc}(\Delta, F) = \frac{\min_{s \in \mathrm{NE}(\Delta, F)} \mathrm{cost}_{sc}(\mathbf{s})}{\mathrm{cost}_{sc}(\mathbf{s}^*_{sc})}$$

$$\mathrm{PoS}_{mc}(\Delta, F) = \frac{\min_{s \in \mathrm{NE}(\Delta, F)} \mathrm{cost}_{mc}(\mathbf{s})}{\mathrm{cost}_{mc}(\mathbf{s}^*_{mc})}$$

When it is clear from the context, we sometimes omit (Δ, F).

2.5 Graph Classes

A *single source single sink directed acyclic graph* is a directed graph with exactly one source node s and sink node t and without cycles. We simply call it a directed acyclic graph (DAG) in this paper.

A *two-terminal series-parallel graph* G is a directed graph with exactly one source node s and sink node t that can be constructed by a sequence of the following operations [4]:

- Create a single directed edge (s, t).
- Given two two-terminal series-parallel graphs G_X with terminals s_X and t_X and G_Y with terminals s_Y and t_Y, form a new graph $S(G_X, G_Y)$ with terminals s and t by identifying $s = s_X$, $t_X = s_Y$ and $t = t_Y$. We call this operation the *series composition* of X and Y.
- Given two two-terminal series-parallel graphs G_X with terminals s_X and t_X and G_Y with terminals s_Y and t_Y, form a new graph $P(G_X, G_Y)$ with terminals s and t by identifying $s = s_X = s_Y$ and $t = t_X = t_Y$. We call this operation the *parallel composition* of G_X and G_Y.

Note that any two-terminal series-parallel graph is a directed acyclic graph. We call a two-terminal series-parallel graph a series-parallel graph (SP graph) for simplicity [4], An SP graph G is a *parallel-link graph* if it is produced by only parallel compositions of single edges.

By the definitions of the above graphs, the following inclusion relation holds:

$$\text{Parallel-link graph} \subseteq \text{SP graph} \subseteq \text{DAG} \subseteq \text{General graph}.$$

3 Capacitated Cost-Sharing Connection Games under Sum-Cost Criterion

In this section, we give tight bounds of PoA and PoS of CSCSG under sum-cost.

3.1 Price of Anarchy (PoA)

Feldman and Ron show that the PoA_{sc} is unbounded on capacitated *undirected* graphs when the cost-sharing function is F_{ord}, that is, $f_e(x_e(\mathbf{s})) = p_e/x_e(\mathbf{s})$ for each $e \in E$ [8]. Because any CSCSG on capacitated undirected graph can be transformed into a CSCSG on capacitated directed graph [6], the PoA_{sc} is unbounded on capacitated directed graphs.

3.1.1 Directed acyclic graphs

In this subsection, we show that the PoA_{sc} of a CSCSG (Δ, F_{ord}) is unbounded even on directed acyclic graphs (DAG). In the proof, we give a directed acyclic graph whose edge costs are represented by some variables. By controlling the variables, we can show that PoA_{sc} can be infinitely large.

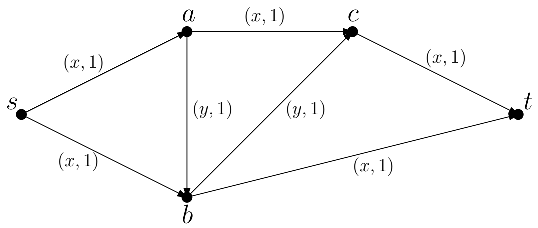

Figure 1. A CSCSG $(\Delta, \{p_e/x_e \mid e \in E\})$ on DAGs such that PoA_{sc} is unbounded. The cost and capacity of an edge e is denoted by (p_e, c_e).

Theorem 1. *There exists a CSCSG* (Δ, F_{ord}) *on DAGs such that* PoA_{sc} *is unbounded.*

Proof. We show an example with two agents such that the PoA_{sc} is unbounded where $x < y$ (see Figure 1). Suppose that one agent uses path $s \to a \to c \to t$, and the other agent uses path $s \to b \to t$. Then the sum-cost of this strategy profile is $5x$.

On the other hand, consider the strategy profile such that one agent uses path $s \to a \to b \to t$ and the other agent uses path $s \to b \to c \to t$. It is easy to see that this strategy profile is a Nash equilibrium and its sum-cost is $4x + 2y$. Thus, the PoA_{sc} is at least $(4x + 2y)/5x = 4/5 + 2y/5x$. By taking $y = x^2$ and x arbitrary large, the PoA_{sc} can be unbounded. □

3.1.2 SP graphs

For SP graphs, we show that PoA_{sc} is at most n, and it is tight.

Feldman and Ron showed that the upper bound of PoA on (undirected) SP graphs is n for the ordinary fair cost-sharing function [8]. We claim that it actually holds for *directed* SP graphs and any cost-sharing functions in $\mathcal{F}^*$. We first introduce Lemma 1 by Feldman and Ron, which holds for *any* payment function. Although the lemma is stated in the game-theoretic context, the claim is essentially about the network flow. Note that the original proof is for undirected SP graphs, but it can be easily modified to directed cases, though we omit the detail.

Lemma 1 ([8]). *Let* $(\Delta, \mathcal{F}_{all})$ *be a CSCSG on SP graphs. For* $r, k \in \mathbb{N}$ *where* $r < k$*, let* $\boldsymbol{s}$ *be a feasible strategy profile of* k *agents, and* $\boldsymbol{s}'$ *be a feasible strategy profile of* r *agents. Then, there is an* s-t *path* s_{r+1} *in* G *that uses only edges used in* $\boldsymbol{s}$ *such that the strategy profile* $(\boldsymbol{s}', s_{r+1})$ *of* $r + 1$ *agents is feasible.*

By using Lemma 1, we obtain the following lemma. Note that Lemma 1 holds for any payment functions, but Lemma 2 holds for any cost-sharing functions in the generalized cost-sharing scheme.

Lemma 2. *Let* $(\Delta, \mathcal{F}^*)$ *be a CSCSG on SP graphs,* *and let $\mathbf{s}^*$ be an optimal strategy profile and $\mathbf{s}$ be a strategy profile that is a Nash equilibrium in $(\Delta, \mathcal{F}^*)$ under sum-cost. Then, the cost of each agent in* $\mathbf{s}$ *is at most* $\mathrm{cost}_{sc}(\mathbf{s}^*)$.

Proof. Let $\mathbf{s}^*$ be an optimal strategy profile under sum-cost and $\mathbf{s}$ be a strategy profile that is a Nash equilibrium. Then we show that the cost of each agent in $\mathbf{s}$ is at most $\mathrm{cost}_{sc}(\mathbf{s}^*)$. For the sake of contradiction, we assume that there is an agent i whose cost $p_i(\mathbf{s})$ is higher than $\mathrm{cost}_{sc}(\mathbf{s}^*)$. Let $\mathbf{s}_{-i}$ be the strategy profile for all agents except for agent i and s_i be the s-t path chosen by agent i in $\mathbf{s}$. Given the strategy profile $\mathbf{s}_{-i}$, there is a feasible s-t path s' that uses only edges in $\mathbf{s}^*$ by Lemma 1. If agent i choose s-t path s' instead of s_i, we claim that the cost of agent i becomes at most $\mathrm{cost}_{sc}(\mathbf{s}^*)$. This can be shown as follows. In the original strategy $\mathbf{s}$, agents using edge e pay $f_e(x_e(\mathbf{s}))$ for each, and in the new strategy $(\mathbf{s}_{-i}, s')$, agent i needs to pay $f_e(x_e(\mathbf{s})+1)$ for $e \in E(s') \setminus E(s_i)$. By taking the summation, the total cost that agent i pays in $(\mathbf{s}_{-i}, s')$ is

$$
\begin{aligned}
&\sum_{e \in E(s') \cap E(s_i)} f_e(x_e) + \sum_{e \in E(s') \setminus E(s_i)} f_e(x_e+1) \\
&\le \sum_{e \in E(s') \cap E(s_i)} p_e + \sum_{e \in E(s') \setminus E(s_i)} p_e \\
&= \sum_{e \in E(s')} p_e \\
&\le \sum_{e \in E(\mathbf{s}^*)} p_e \\
&\le \mathrm{cost}_{sc}(\mathbf{s}^*).
\end{aligned}
$$

The first and last inequalities come from properties (3') and (2') of our generalized cost-sharing scheme, respectively. Thus, agent i can pay lower cost by deviating to this path. This contradicts that the strategy profile $\mathbf{s}$ is a Nash equilibrium. Thus, $p_j(\mathbf{s}) \le \mathrm{cost}_{sc}(\mathbf{s}^*)$ for any agent j. □

By Lemma 2, we can see that the total cost of the agents in a Nash equilibrium is at most $n \cdot \mathrm{cost}_{sc}(\mathbf{s}^*)$, which implies the following.

Lemma 3. *In CSCSG $(\Delta, \mathcal{F}^*)$ on SP graphs,* PoA_{sc} *is at most n.*

As for the lower bound of PoA_{sc}, Anshelevich et al. gave an example of *uncapacitated* cost-sharing connection games on parallel-link graphs where PoA_{sc} is n [1]. The example is a game of n agents on a parallel-link graph consisting of two vertices and two directed edges whose costs are defined by 1 and n, respectively. Since a CSCSG such that the capacity of each edge is n is equivalent to an uncapacitated cost-sharing connection game, we obtain the same lower bound for CSCSG.

Lemma 4. *There exists a CSCSG $(\Delta, F_{\mathrm{ord}})$ where* PoA_{sc} *is n even on parallel-link graphs.*

By Lemmas 3 and 4, we obtain Theorem 2.

Theorem 2. *For any CSCSG $(\Delta, \mathcal{F}^*)$,* PoA_{sc} *is at most n. Furthermore, there exists a CSCSG with* $\mathrm{PoA}_{sc} = n$ *on parallel-link graphs.*

3.2 Price of Stability (PoS)

We show that PoS_{sc} in $(\Delta, \mathcal{F}^*)$ on SP graphs is at most n and it is tight.

Lemma 5. *For any CSCSG $(\Delta, \mathcal{F}^*)$,* PoS_{sc} *is at most n.*

Proof. Let $\mathbf{s}^*$ be an optimal strategy profile under sum-cost. Consider agents repeatedly deviate from $\mathbf{s}^*$ to reduce their costs. Eventually, this procedure results in a Nash equilibrium $\mathbf{s}$ by Proposition 1.

Recall that the change $\Phi(\mathbf{s}) - \Phi(s'_j, \mathbf{s}_{-j})$ from $\mathbf{s}$ to a new strategy profile $(s'_j, \mathbf{s}_{-j})$ equals the change of the cost of agent j [11]. Thus, $\Phi(\mathbf{s}) \le \Phi(\mathbf{s}^*)$ holds.

By property (2') of our generalized cost-sharing scheme, for any edge $e \in E$ and strategy profile $\mathbf{s}$, $\sum_{x=1}^{x_e(\mathbf{s})} f_e(x) \le n p_e$ holds. Then we transform the potential function $\Phi(\mathbf{s}^*)$ for strategy profile $\mathbf{s}^*$ as follows:

$$
\begin{aligned}
\Phi(\mathbf{s}^*) &= \sum_{e \in E} \sum_{x=1}^{x_e(\mathbf{s}^*)} f_e(x) = \sum_{e \in E(\mathbf{s}^*)} \sum_{x=1}^{x_e(\mathbf{s}^*)} f_e(x) \\
&\le \sum_{e \in E(\mathbf{s}^*)} n p_e \le n \cdot \mathrm{cost}_{sc}(\mathbf{s}^*). \qquad (1)
\end{aligned}
$$

Because $\mathrm{cost}_{sc}(\mathbf{s}) \le \Phi(\mathbf{s})$ holds, we have $\mathrm{cost}_{sc}(\mathbf{s}) \le \Phi(\mathbf{s}) \le \Phi(\mathbf{s}^*) \le n \cdot \mathrm{cost}_{sc}(\mathbf{s}^*)$. Therefore, $\mathrm{PoS}_{sc}(\Delta, \mathcal{F}^*) \le n \cdot \mathrm{cost}_{sc}(\mathbf{s}^*)/\mathrm{cost}_{sc}(\mathbf{s}^*) = n$. □

Lemma 6. *There exists a CSCSG $(\Delta, \mathcal{F}^*)$ with* $\mathrm{PoS}_{sc} = n$ *on parallel-link graphs.*

Proof. Consider the following CSCSG (Δ, F) with n agents on the parallel-rink graph with $n+1$ edges $e_0, \ldots, e_n$, illustrated in Figure 2. Let δ be a positive constant where $0 < \delta < 1/n$. The function $g(x)$ is defined as $g(x) = \delta(-n + x - 1) + 1$. The cost and the capacity of edge e_i are $g(i)$ and 1, respectively, for $1 \le i \le n-1$. Also, the cost and the capacity of edge e_n is $1+\epsilon$ and n, respectively. Finally, we define the cost-sharing function of edge e as $f_e(x) = (-\delta x + \delta + 1)p_e$. Note that f_e satisfies $f_e(1) = p_e$ and $f_e(x) \ge p_e/x$ for $0 < \delta < 1/n$, and thus f_e's belong to $\mathcal{F}^*$.

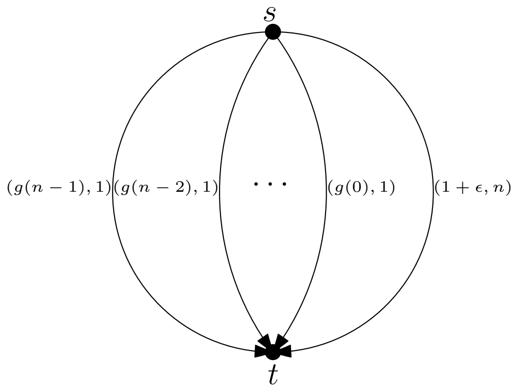

Figure 2. A CSCSG $(\Delta, \mathcal{F}^*)$ with $\mathrm{PoS}_{sc} = n$ on a parallel-link. The cost and capacity of edge e is denoted by (p_e, c_e).

Let $\mathbf{s}^*$ be a strategy profile where every agent uses edge e_n. The sum-cost of $\mathbf{s}^*$ is $\mathrm{cost}_{sc}(\mathbf{s}^*) = (-\delta n + \delta + 1)(1+\epsilon)$.

Next, let $\mathbf{s}$ be a strategy profile where agent i uses edge e_{i-1}. We show that strategy profile $\mathbf{s}$ is a unique Nash equilibrium. Suppose that $n-k$ agents use edge e_n and k agents use some edges in $\{e_0, \ldots, e_{n-1}\}$. Then the cost of e_n is as follows: $f_{e_n}(n-k) = (-\delta(n-k) + \delta + 1)(1+\epsilon) = (\delta(-n+k+1)+1)(1+\epsilon) = g(k)(1+\epsilon)$. Note that $f(n) = g(0)(1+\epsilon)$. Therefore, for any $0 \leq k \leq n-1$, there is an agent that moves from e_n to an edge in $\{e_0, \ldots, e_{n-1}$. Because the capacity of each e_i for $0 \leq i \leq n-1$ is 1, $\mathbf{s}$ is a unique Nash equilibrium.

The sum-cost in strategy profile $\mathbf{s}$ is as follows:

$$\begin{aligned}\mathrm{cost}_{sc}(\mathbf{s}) &= \sum_{k=0}^{n-1} g(k) = \sum_{k=0}^{n-1} (\delta(-n+k-1)+1) \\ &= \delta\left(-n^2 + \frac{1}{2}n(n-1) - n\right) + n \\ &= -\frac{\delta}{2}\left(n^2+3n\right) + n.\end{aligned}$$

Set $\delta = 1/n^3$. Then,

$$\begin{aligned}\mathrm{PoS}_{sc}(\Delta, \mathcal{F}^*) \geq \frac{\mathrm{cost}_{sc}(\mathbf{s})}{\mathrm{cost}_{sc}(\mathbf{s}^*)} &= \frac{-\frac{\delta}{2}\left(n^2+3n\right)+n}{(-\delta n + \delta + 1)(1+\epsilon)} \\ &= \frac{-\frac{1}{2}\left(\frac{1}{n}+\frac{3}{n^2}\right)+n}{(-\frac{1}{n^2}+\frac{1}{n^3}+1)(1+\epsilon)}.\end{aligned}$$

When n is arbitrary large and ϵ is arbitrary small, $\mathrm{PoS}_{sc}(\Delta, \mathcal{F}^*)$ becomes n. □

Theorem 3. *For any CSCSG $(\Delta, \mathcal{F}^*)$ on SP graphs, PoS_{sc} is at most n. Furthermore, there is a CSCSG $(\Delta, \mathcal{F}^*)$ with $\mathrm{PoS}_{sc} = n$.*

4 Capacitated Cost-Sharing Connection Games under Max-Cost Criterion

In this section, we give the tight bounds of PoA and PoS of CSCSG under max-cost.

4.1 Price of Anarchy (PoA)

Feldman et al. show that PoA_{mc} is unbounded on undirected graphs [8]. As with the sum-cost case, we can also show that PoA_{mc} is unbounded on a directed graph by using the transformation in [6].

4.1.1 Directed acyclic graphs

We show that PoA_{mc} of an SCSCG $(\Delta, F_{\mathrm{ord}})$ is unbounded even on DAGs.

Theorem 4. *There exists an SCSCG $(\Delta, F_{\mathrm{ord}})$ on DAGs such that PoA_{mc} is unbounded.*

Proof. We show that the PoA_{mc} of the SCSCG illustrated in Figure 1 is unbounded. The strategy is the same as the sum-cost case. The optimal cost is $3x$, and the maximum cost of all Nash equilibrium is $2x+y$. Thus, we have $\mathrm{PoA}_{mc} = (2x+y)/3x = 2/3 + y/3x$. Because x and y are arbitrary where $x < y$, PoA_{mc} can be unbounded. □

4.1.2 SP graphs

For any CSCSG $(\Delta, \mathcal{F}^*)$, we show that PoA_{mc} on SP graph is at most n.

Lemma 7. *For any CSCSG $(\Delta, \mathcal{F}^*)$ on SP graphs, PoA_{mc} is at most n.*

Proof. Let $\mathbf{s}^*$ be an optimal strategy profile under max-cost. Then for any Nash equilibrium $\mathbf{s}$, it holds that $\mathrm{cost}_{mc}(\mathbf{s}) \leq \mathrm{cost}_{sc}(\mathbf{s}^*) \leq n \cdot \mathrm{cost}_{mc}(\mathbf{s}^*)$. The first inequality holds by Lemma 2. By the definition of max-cost, the second inequality holds. Thus, we have $\mathrm{PoA}_{mc}(\Delta, \mathcal{F}^*) \leq n \cdot \mathrm{cost}_{mc}(\mathbf{s}^*)/\mathrm{cost}_{mc}(\mathbf{s}^*) = n$. □

On the other hand, we observe that the PoA_{mc} of the game used in Lemma 4 is n. Therefore, we obtain Theorem 5 as follows.

Theorem 5. *For any CSCSG $(\Delta, \mathcal{F}^*)$ on SP graphs, PoA_{mc} is at most n. Furthermore, there is a CSCSG with $\mathrm{PoA}_{mc} = n$ even on parallel-link graphs.*

4.2 Price of Stability (PoS)

For the lower bound, Feldman and Ron showed that PoS_{mc} is n on undirected parallel-link graphs. By orienting all the edges from a vertex to the other vertex in an undirected parallel-link graph, we can see that PoS_{mc} is n on a directed parallel-link graph.

In the following, we show that PoS_{mc} is at most n.

Lemma 8. *For any CSCSG* $(\Delta, \mathcal{F}^*)$*,* PoS_{mc} *is at most* n*.*

Proof. The outline of the proof follows that of [6, Theorem 3], though we extend it to our generalized setting. Let $\mathbf{s}^*$ be an optimal strategy profile under max-cost. As with Lemma 5, consider agents repeatedly deviate from $\mathbf{s}^*$ to reduce their costs. By Proposition 1, we obtain a Nash equilibrium $\mathbf{s}$ in the end of the deviations. Without loss of generality, we can scale the edge costs so that $\mathrm{cost}_{sc}(\mathbf{s}^*) = n$, and as the result, we have $\mathrm{cost}_{mc}(\mathbf{s}^*) \geq 1$.

If $\mathrm{cost}_{mc}(\mathbf{s}) \leq n$, then we obtain $\mathrm{PoS}_{mc}(\Delta, \mathcal{F}^*) \leq n$. Otherwise, the following inequality holds:

$$n < \mathrm{cost}_{mc}(\mathbf{s}) \leq \mathrm{cost}_{sc}(\mathbf{s}) \leq \Phi(\mathbf{s}) < \Phi(\mathbf{s}^*).$$

Note that the third inequality comes from that $\mathcal{F}^*$'s functions are non-increasing and the forth inequality comes from the definition of the potential function. For some $\alpha, \beta, \delta > 0$, let $\Phi(\mathbf{s}^*) = n + \alpha$, $\mathrm{cost}_{mc}(\mathbf{s}) = n + \beta$, and $\Phi(\mathbf{s}) = \Phi(\mathbf{s}^*) - \delta$. Note that $0 < \beta \leq \alpha - \delta$.

Let $\mathbf{s}_{-i}$ be the strategy profile of $n-1$ agents except for agent i who pays $\mathrm{cost}_{mc}(\mathbf{s})$. Because the change of the potential function equals the change of the cost of an agent who deviates, we have:

$$\begin{aligned}\Phi(\mathbf{s}_{-i}) = \Phi(\mathbf{s}) - \mathrm{cost}_{mc}(\mathbf{s}) &= (n + \alpha - \delta) - (n + \beta) \\ &= \alpha - \beta - \delta.\end{aligned}$$

We construct a strategy profile $\mathbf{s}'$ of n agents by combining $\mathbf{s}^*$ and $\mathbf{s}_{-i}$ as follows. We define $\bar{G} = (V, \bar{E})$ where $\bar{E} = E(\mathbf{s}^*) \cup E(\mathbf{s}_{-i})$ and $\bar{c}(e) = \max\{x_e(\mathbf{s}^*), x_e(\mathbf{s}_{-i})\}$ as a directed and capacitated graph. Now, we have $n-1$ paths in the strategy profile $\mathbf{s}_{-i}$ and n paths in the strategy profile $\mathbf{s}^*$. Here, we regard the strategy profile $\mathbf{s}_{-i}$ as s-t flow and let $\bar{G}_{\mathbf{s}_{-i}}$ be the residual network of $\bar{G}$ for $\mathbf{s}_{-i}$. Since $\bar{G}$ admits flow with size n while the size of flow constructed from $\mathbf{s}_{-i}$ is $n-1$, there is an augmenting s-t path s' in $\bar{G}_{\mathbf{s}_{-i}}$. By adding the augmenting path in $\bar{G}_{\mathbf{s}_{-i}}$ to $\mathbf{s}_{-i}$, we construct the strategy profile $\mathbf{s}' = (\mathbf{s}_{-i}, s')$. We notice that the addition increases the number of agents that use $e \in \bar{E}(s')$ by at most 1 and $x_e(\mathbf{s}^*) > 0$ for every edge $e \in \bar{E}(s')$.

Let $p(s') = \sum_{e \in \bar{E}(s')} p_e$ be the cost of path s'. By agent i choosing s', the potential increases by at most p_e for each $e \in \bar{E}(s')$ due to property (3') of $\mathcal{F}^*$, which implies that the total increase is at most $p(s')$; we have $\Phi(\mathbf{s}') \leq \Phi(\mathbf{s}_{-i}) + p(s')$. Moreover, since $x_e(\mathbf{s}^*) > 0$ for every edge $e \in \bar{E}(s')$, we obtain $p(s') = \sum_{e \in \bar{E}(s')} p_e \leq \mathrm{cost}_{sc}(\mathbf{s}^*) = n$ by property (2') of $\mathcal{F}^*$. Hence, we have

$$\Phi(\mathbf{s}') \leq (\alpha - \beta - \delta) + n = \Phi(\mathbf{s}) - \beta < \Phi(\mathbf{s}).$$

Let $\mathbf{s}''$ be a Nash equilibrium obtained from $\mathbf{s}'$ by the deviations of agents. We then have $\Phi(\mathbf{s}'') \leq \Phi(\mathbf{s}') < \Phi(\mathbf{s})$.

If $\mathrm{cost}_{mc}(\mathbf{s}'') \leq n$, then $\mathrm{PoS}_{mc}(\Delta, \mathcal{F}^*) \leq n$ due to $\mathrm{cost}_{mc}(\mathbf{s}^*) \geq 1$. Thus, we are done. Otherwise, we repeat the above procedure starting from $\mathbf{s}''$ instead of $\mathbf{s}$. For each time, if we obtain a Nash equilibrium with max-cost higher than n, that Nash equilibrium has strictly less potential than the previous Nash equilibrium. Since the number of strategy profiles is finite, we eventually obtain a Nash equilibrium whose max-cost is at most n. Therefore, $\mathrm{PoS}_{mc}(\Delta, \mathcal{F}^*) \leq n$ holds. □

Theorem 6. *For any CSCSG* $(\Delta, \mathcal{F}^*)$*,* PoS_{mc} *is at most* n*, and it is tight.*

5 Asymmetric Games

In this section, we consider the *capacitated asymmetric cost-sharing connection game (CACSG)*, where agents have different source and sink nodes.

Because the CACSG is a generalization of CSCSG, the lower bound of CSCSG holds for the CACSG.

Theorem 7. *For any CACSG* $(\Delta, \mathcal{F}^*)$*,* PoA_{sc} *and* PoA_{mc} *are unbounded even on DAGs.*

As for the sum-cost, it is easily seen that Lemma 5 holds for the asymmetric case.

Theorem 8. *For any CACSG* $(\Delta, \mathcal{F}^*)$*,* PoS_{sc} *is at most* n *and it is tight.*

For the max-cost, we show that PoS_{mc} is at most n^2.

Theorem 9. *For any CACSG* $(\Delta, \mathcal{F}^*)$*,* PoS_{mc} *is at most* n^2*.*

Proof. Let $\mathbf{s}^*$ be an optimal strategy profile under max-cost and $\mathbf{s}$ be a Nash equilibrium obtained from $\mathbf{s}^*$ by the deviations of agents. By the definition of the potential function, $\Phi(\mathbf{s}) \leq \Phi(\mathbf{s}^*)$ holds. By Eq. (1) in Lemma 5, $\Phi(\mathbf{s}^*) \leq n \cdot \mathrm{cost}_{sc}(\mathbf{s}^*)$ holds. Because $\mathrm{cost}_{sc}(\mathbf{s}^*)/n \leq \mathrm{cost}_{mc}(\mathbf{s}^*)$, we have $\Phi(\mathbf{s}) \leq n^2 \cdot \mathrm{cost}_{mc}(\mathbf{s}^*)$. Finally, since $\mathrm{cost}_{mc}(\mathbf{s}) \leq \Phi(\mathbf{s})$, $\mathrm{cost}_{mc}(\mathbf{s}) \leq n^2 \cdot \mathrm{cost}_{mc}(\mathbf{s}^*)$ holds. Thus, we have: $\mathrm{PoS}_{mc} \leq n^2 \cdot \mathrm{cost}_{mc}(\mathbf{s}^*)/\mathrm{cost}_{mc}(\mathbf{s}^*) = n^2$. □

For the lower bound of PoS_{mc}, Erlebach and Radoja showed that there is a CACSG $(\Delta, F_{\mathrm{ord}})$ with $\mathrm{PoS}_{mc} = \Omega(n \log n)$ [6]. However, there is a gap between $n \log n$ and n^2 with respect to PoS_{mc}.

6 Conclusion

In this paper, we studied the capacitated symmetric / asymmetric cost-sharing connection game (CSCSG, CACSG) under the generalized cost-sharing scheme that models more realistic scenarios such as the situation where the overhead costs arise. In particular, we investigated the games from the viewpoint of PoA and PoS under two types of social costs: sum-cost and max-cost. All the bounds that we give for CSCSG are tight. In spite of the generalization, all the bounds under the ordinary cost-sharing function still hold with one exception; for PoS of sum-cost, we found a substantial difference between the ordinary cost-sharing function and the generalized scheme, where the former is $\log n$ and the latter is n.

In this paper, we mainly focused on CSCSG. For the asymmetric games, however, there is still a gap for PoS under max-cost. Thus, filling the gap is an interesting open problem. Moreover, it would be worth to consider some other concepts of stability, such as Strong Price of Anarchy (SPoA) and Strong Price of Stability (SPoS) to CSCSG and CACSG under the generalized cost-sharing scheme.

Reference

[1] Elliot Anshelevich, Anirban Dasgupta, Jon Kleinberg, Éva Tardos, Tom Wexler, and Tim Roughgarden. The price of stability for network design with fair cost allocation. *SIAM Journal on Computing*, 38(4):1602–1623, 2008.

[2] C. Chau and K. Elbassioni. Quantifying inefficiency of fair cost-sharing mechanisms for sharing economy. *IEEE Transactions on Control of Network Systems*, 5(4):1809–1818, 2018.

[3] Sid Chi-Kin Chau, Jiajia Xu, Wilson Bow, and Khaled Elbassioni. Peer-to-peer energy sharing: Effective cost-sharing mechanisms and social efficiency. In *Proceedings of the Tenth ACM International Conference on Future Energy Systems*, page 215225, 2019.

[4] David Eppstein. Parallel recognition of series-parallel graphs. *Information and Computation*, 98:41–55, 1992.

[5] Amir Epstein, Michal Feldman, and Yishay Mansour. Strong equilibrium in cost sharing connection games. *Games and Economic Behavior*, 67(1):51–68, 2009.

[6] Thomas Erlebach and Matthew Radoja. Further results on capacitated network design games. In *Proceedings of the 8th Symposium on Algorithmic Game Theory (SAGT 2015)*, pages 57–68, 2015.

[7] Michal Feldman and Geri Ofir. Do capacity constraints constrain coalitions? *ACM Transactions on Economics and Computation*, 5(1):No.8, 2016.

[8] Michal Feldman and Tom Ron. Capacitated network design games. *Theory of Computing Systems*, 57:576597, 2015.

[9] Alfredo Goldman and Carlos Queiroz. A model for parallel job scheduling on dynamical computer grids. *Concurrency and Computation: Practice and Experience*, 16(5):461–468, 2004.

[10] Anders Hansen Henten and Iwona Maria Windekilde. Transaction costs and the sharing economy. *info*, 18(1):1–15, 2016.

[11] Dov Monderer and Lloyd S. Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.

[12] Ievgen Redko and Charlotte Laclau. On fair cost sharing games in machine learning. In *Proceedings of the 33rd AAAI Conference on Artificial Intelligence (AAAI 2019)*, pages 4790–4797, 2019.

[13] Vasilis Syrgkanis. The complexity of equilibria in cost sharing games. In *Proceedings of the 6th Workshop on Internet and Network Economics (WINE 2010)*, pages 366–377, 2010.